Attachment

Definition of the Triangular Grid Based on an Icosahedron

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A triangular grid based on an icosahedron was first introduced in a meteorological model by **Sadourny et al.** (1968) and **Williamson** (1969). The approach outlined here, especially the code implementation, is based on the work of **Baumgardner** (1995).

To construct the triangular grid based on an icosahedron, the unit-sphere, i.e. a sphere with radius 1, is divided into 20 spherical triangles of equal size by placing a plane icosahedron into the sphere (Fig. 1). The 12 vertices of the icosahedron touch the sphere, one vertex coincide with the north pole (NP), the opposite one with the south pole (SP), for simplicity.

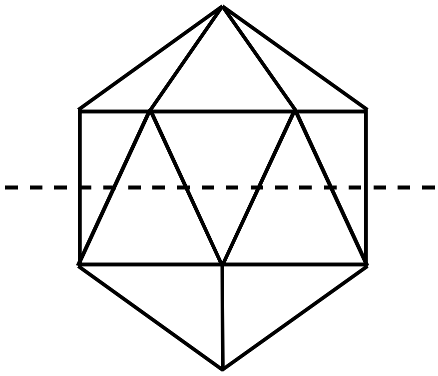

Figure 1 Plane icosahedron consisting of 20 plane triangles

The 12 vertices are connected by great circles to form 20 **main spherical triangles**. Since each of the 12 vertices is surrounded by 5 main spherical triangles (Fig. 2) the angles between two sides of the main triangles are $2\pi/5$ or 72°.

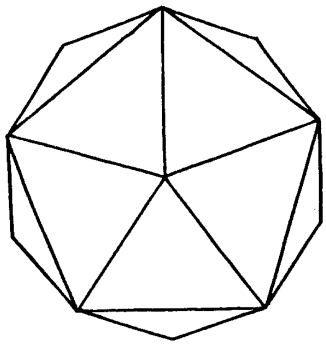

Figure 2 The five main spherical triangles at the north pole

The length w of a main triangle side follows from Fig. 3 and equation (1)

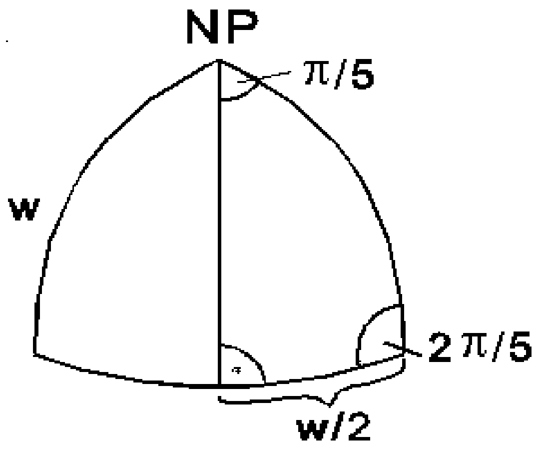

Figure 3 One main spherical triangle at the north pole

$$\cos\frac{1}{2}\,w = \frac{\cos\frac{\pi}{5}}{\sin 2\frac{\pi}{5}} = \frac{1}{2\sin\frac{\pi}{5}} \tag{1}$$

Thus w ~ 1.107149. On the unit-sphere, w is identical to $\pi/2 - \varphi$ with the latitude φ of the lower corner of the triangle. Thus w is a measure of the latitude of the lower vertices of the triangle in Fig. 3.

Two adjacent main spherical triangles are combined to form a "**diamond**", i.e. a logical square block. Five of the diamonds originate from the north pole, five from the south pole. The numbering and order of the diamonds are outlined in Fig. 4.

The diamonds 1 to 5 are the "northern" ones, i.e. they start at the north pole, diamonds 6 to 10 start at the south pole. The so called home vertex of each diamond (in the order 1, 6, 2, 7, 3, 8, 4, 9, 5, 10) is shifted by $\pi/5$ to the east starting at $-\pi/5$ for the first diamond. Thus the 10 home vertices have the following geographical co-ordinates (λ and φ) on the unit-sphere (Table 1).

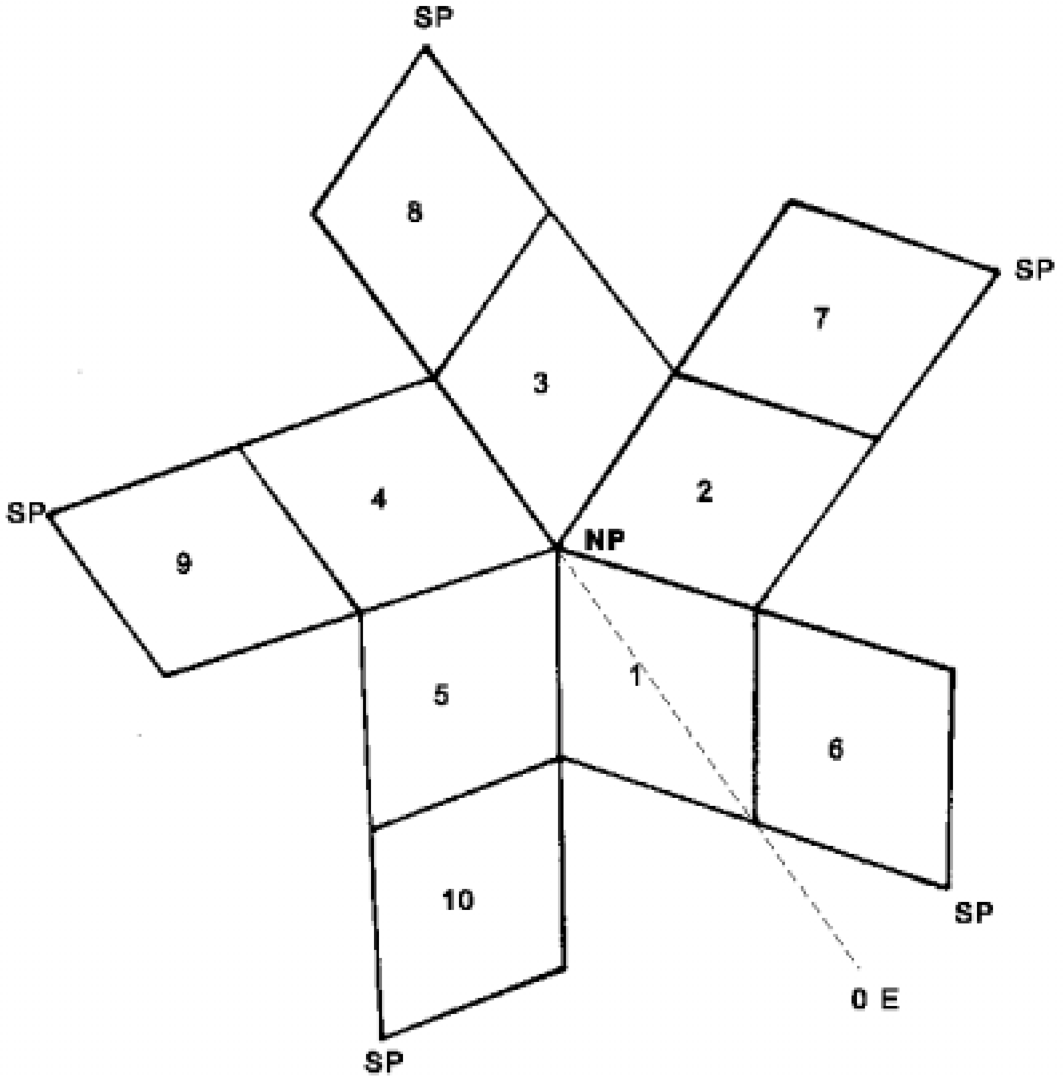

Figure 4 The 20 main spherical triangles combined to 10 diamonds

Table 1 Geographical co-ordinates (λ and φ) of the home vertices of the 10 diamonds

Diamond #	1	2	3	4	5
λ	$-\pi/5$	$\pi/5$	$3\pi/5$	$5\pi/5$	$-3\pi/5$
φ	$\pi/2 - w$	$\pi/2 - w$	$\pi/2 - w$	$\pi/2 - w$	$\pi/2 - w$

Diamond #	6	7	8	9	10
λ	0	$2\pi/5$	$4\pi/5$	$-4\pi/5$	$-2\pi/5$
φ	$w - \pi/2$	$w - \pi/2$	$w - \pi/2$	$w - \pi/2$	$w - \pi/2$

A Cartesian co-ordinate system is placed into the unit-sphere with the origin in the centre of the sphere, the z-axis towards the north pole and the x-axis into the direction of the Greenwich meridian. The Cartesian co-ordinates (x, y, z) of a point on the unit-sphere follow from (2).

$$
\begin{aligned}
x &= \cos\lambda \cos\varphi = \cos\lambda \sin w \\
y &= \sin\lambda \cos\varphi = \sin\lambda \sin w \\
z &= \sin\varphi = \cos w
\end{aligned}
\tag{2}
$$

Thus the two pole vertices have the Cartesian co-ordinates (0, 0, 1) and (0, 0, -1), respectively.

The geographical co-ordinates (λ, φ) of a point on the unit-sphere with the Cartesian co-ordinates (x, y, z) follow from (3) which may be derived from (2).

$$
\begin{aligned}
\lambda &= \arctan \frac{y}{x} \\
\varphi &= \arcsin z
\end{aligned}
\tag{3}
$$

For the grid generation, the sides w of the 20 main triangles are iteratively subdivided into **ni** equal parts to form sub-triangles. Each point in a main triangle is now surrounded by six triangles (Fig. 5) and is, therefore, in the centre of a hexagon (See also Fig. 6). However, the points which form the vertices of the icosahedron are surrounded by only five triangles and therefore these 12 special points are the centres of pentagons. For the first subdivision, w may be divided into **three** parts, later on, only **bisections** are allowed. This restriction is due to the use of a multigrid solver (MG) for the Helmholtz-equations in the semi-implicit time stepping. MG-solvers work efficiently with such mesh refinements. Thus the number **ni** of subdivisions of w is factorized according to (4).

$$ni = 3^{n3}\, 2^{n2} \tag{4}$$

with n3 = 0 or 1 and n2 $\geq$ 0.

Fig. 5 shows the resulting grids for ni = 1, 2, 4 and 8, i.e. n2 = 0, 1, 2, 3 with n3=0.

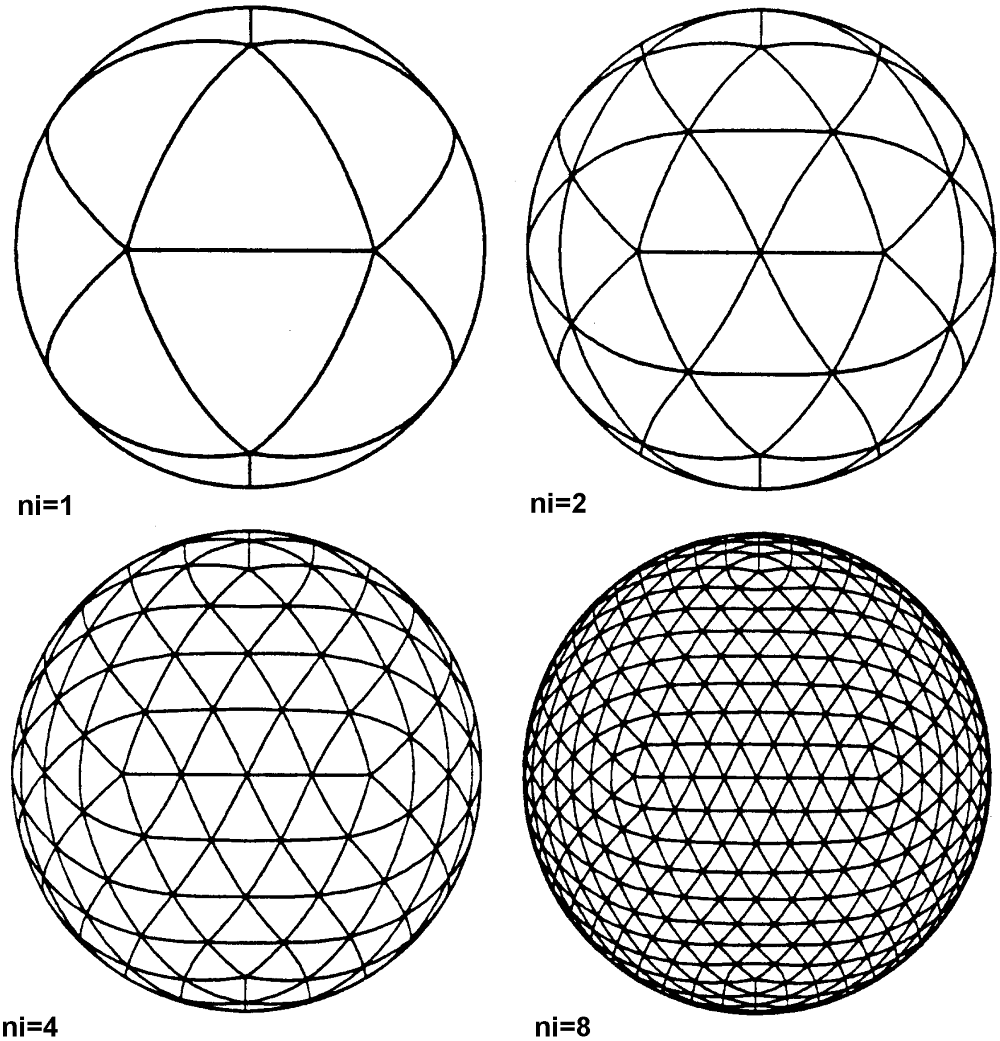

Figure 5 **Spherical triangular grids for different values ni of the subdivision of the main spherical triangles**

The model grid-points (nodes) are located at the vertices of the triangles; thus there are $(ni+1)^2$ grid-points within one diamond. Of these $(ni+1)^2$ grid-points, ni*ni are "uniquely" identified with each diamond, one extra row and column is shared between neighbouring diamonds.

On the earth with a mean radius R_E = 6371229 m the length L of a side of a main triangle is L = w R_E = 7053898 m. The mesh size Δ of the triangular grid with ni equal intervals on the side of a main triangle is not constant within a diamond but varies by 20 % at most on the sphere and is approximately given by (5). E.g., for ni = 32, Δ varies between 220 and 263 km, for ni = 64, Δ varies between 110 and 132 km, and for ni = 128, Δ varies between 55 and 66 km.

$$\Delta \approx \frac{w\ R_E}{ni} \tag{5}$$

The number N of grid-points, not counting the common edges of the diamond, is given by (6).

$$N = 10\ ni^2 + 2 \tag{6}$$

Tab. 2a gives the mesh size Δ, the number N of grid-points and the time step Δt for different values of ni, if only bisections are performed, i.e. ni = 2^{n2}. The time step Δt is calculated under the assumption that an air parcel does not leave the region of the 6 surrounding triangles during the period of twice the time step, i.e. 2 Δt < h/v_{Max} with the height h of the spherical triangle (which is the shortest distance for leaving a triangle) and v_{Max} the maximum wind speed ($\approx$125 m/s) assuming that the fast gravity waves are treated semi-implicitly. The height h of a spherical triangle approximately follows from (7) and is about 5% smaller than the mesh size Δ.

$$h \approx \arcsin\left(\sin\frac{w}{ni}\sin\frac{2\pi}{5}\right) R_E \tag{7}$$

Table 2a Mesh size Δ, height h, number N of grid-points and time step Δt for the spherical triangular mesh using only bisections

ni	16	32	64	128	256
Δ (km)	441	220	110	55	28
h (km)	420	210	105	52	26
N	2 562	10 242	40 962	163 842	655 362
Δt (s)	1 600	800	400	200	100

Table 2b Mesh size Δ, height h, number N of grid-points and time step Δt for the spherical triangular mesh using first a trisection followed by bisections

ni	12	24	48	96	192
Δ (km)	588	294	147	73	37
h (km)	559	279	140	69	35
N	1 442	5 762	23 042	92 162	368 642
Δt (s)	2 200	1 100	550	275	138

Each grid-point is representative for a spherical polygon with six vertices (Fig. 6) except the 12 vertices of the icosahedron which are surrounded by five triangles only.

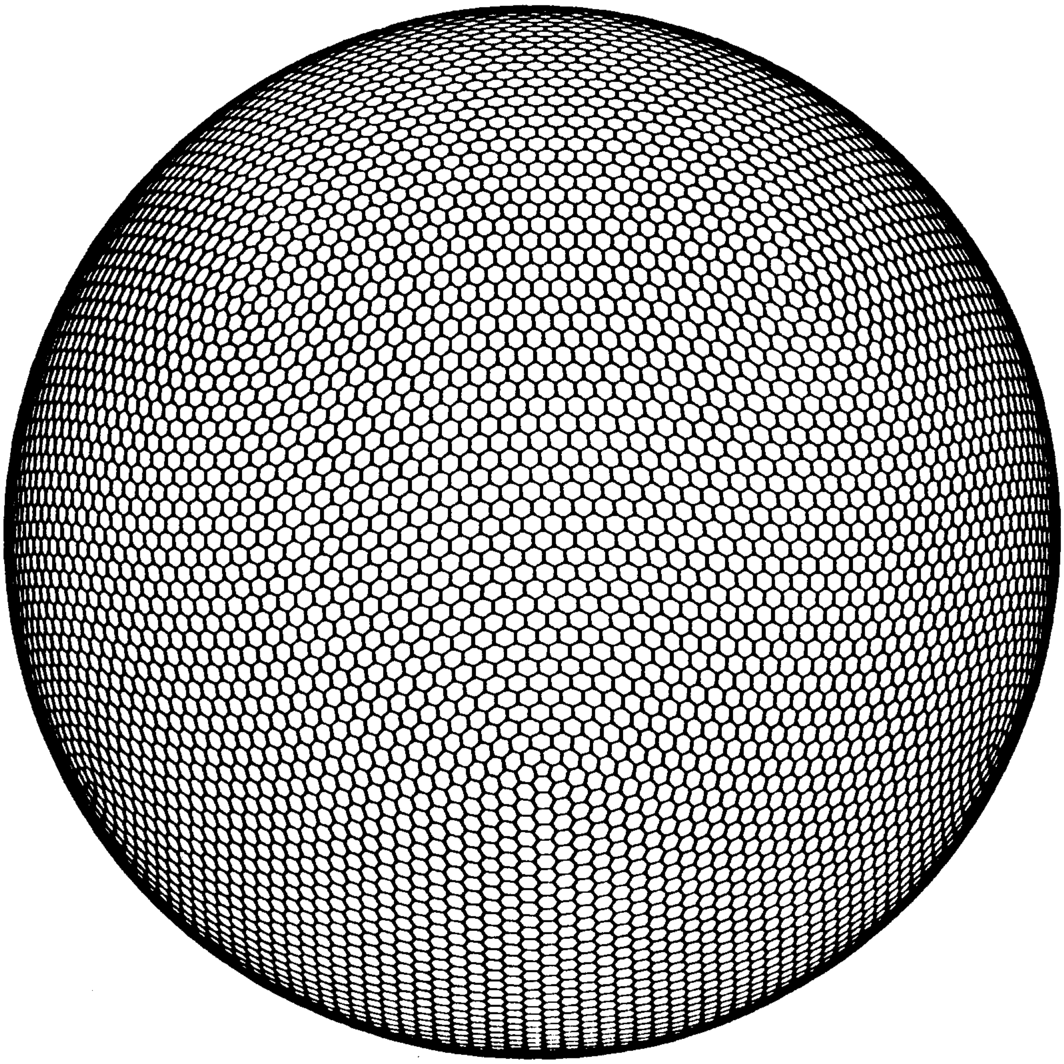

Figure 6 Polygons which represent the area of representativeness of a triangular grid-point

The grid-point indices are defined in the following way (Fig. 7).

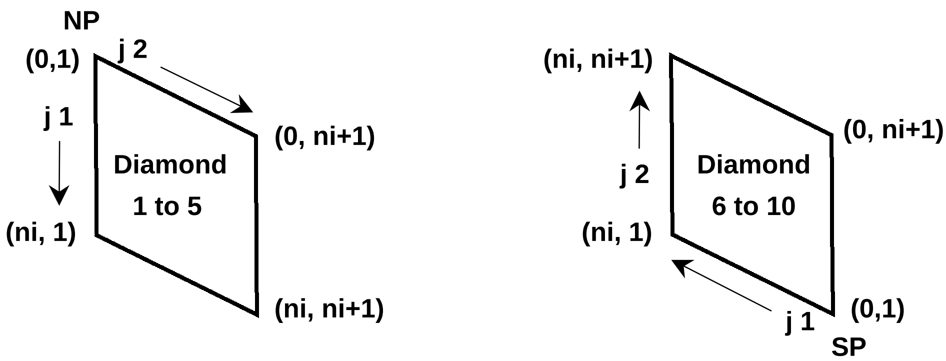

Figure 7 Grid-point indices for a northern (left) and southern (right) diamond.

The start address (0, 1) reflects the philosophy that the ni*ni grid-points which are "uniquely" identified within each diamond have the indices 1 to ni for rows and columns. The extra row and column needed for communication between neighbouring diamonds is lying in one case at the beginning of the first co-ordinate and in the other case at the end of the second. Thus points outside the range (1:ni, 1:ni) are belonging to the neighbouring diamonds and have to be communicated during each time step. Grid-point (0, 1), respectively is the north pole for the diamonds 1 to 5, and the south pole for the diamonds 6 to 10.

The calculation of the subdivision of the great circle between two points P_1 (with the location vector $\mathbf{x}_1$) and P_2 (with location vector $\mathbf{x}_2$) can be derived from Fig. 8.

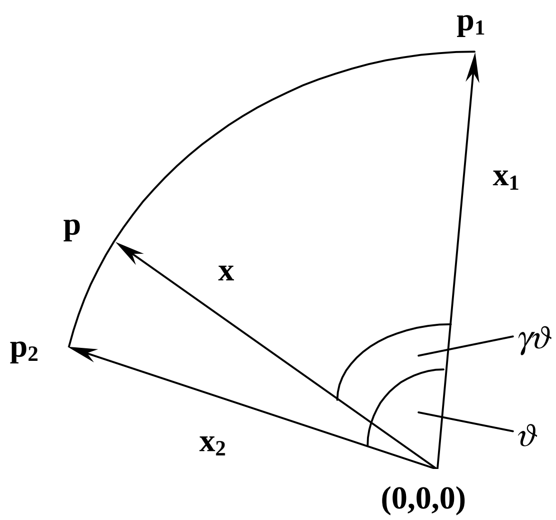

Figure 8 Calculation of the subdivision of the great circle through the points P_1 and P_2 on the unit-sphere.

Since $\mathbf{x}_1$ and $\mathbf{x}_2$ define the great circle plane through P_1 and P_2, all points P with the location vector $\mathbf{x}$ on the great circle may be written as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$.

$$\mathrm{x} = \alpha\, \mathrm{x}_1 + \beta\, \mathrm{x}_2 \tag{8}$$

The coefficients α and β are derived from the condition that $\mathbf{x}$ is a vector on the unit-sphere and the angle between $\mathbf{x}$ and $\mathbf{x}_1$ is given by $\gamma\vartheta$ with γ between 0 and 1 and ϑ the angle between $\mathbf{x}_1$ and $\mathbf{x}_2$, i.e. the length of the great circle between P_1 and P_2.

$$\begin{aligned} \mathrm{x} * \mathrm{x} &= 1 = \alpha^2 + \beta^2 + 2\alpha\,\beta\cos\vartheta \\ \mathrm{x} * \mathrm{x}_1 &= \cos(\gamma\,\vartheta) = \alpha + \beta\cos\vartheta \end{aligned} \tag{9}$$

Substituting α from the second equation into the first one, the coefficients follow from (10).

$$\begin{aligned} \alpha &= \frac{\sin((1-\gamma)\,\vartheta)}{\sin\vartheta} \\ \beta &= \frac{\sin(\gamma\,\vartheta)}{\sin\vartheta} \end{aligned} \tag{10}$$

The angle ϑ between $\mathbf{x}_1$ and $\mathbf{x}_2$ follows from the scalar product $\mathbf{x}_1 * \mathbf{x}_2$ or by calculating the distance d between $\mathbf{x}_1$ and $\mathbf{x}_2$ and observing that sin $\vartheta/2 = d/2$.

The grid-point co-ordinates (x, y, z) of all triangle vertices on the unit-sphere are derived with (8) using the coefficients of (10).

The $(ni+1)^2$ grid-points in a diamond form the vertices of 2 ni^2 triangles (Fig. 9) half of those point northward, half southward.

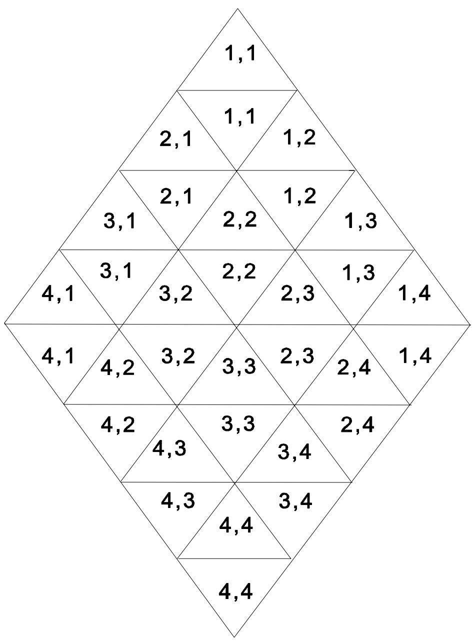

Figure 9 The 2 ni^2 triangles in a diamond defined by the (ni+1)^2 vertices for ni = 4

To calculate the co-ordinates (x_c, y_c, z_c) of the triangle centres P_c, the co-ordinates of the three triangles vertices P_1, P_2 and P_3 are summed and normalized.

$$x_c = (x_1 + x_2 + x_3)\; x_N$$
$$y_c = (y_1 + y_2 + y_3)\; x_N$$
$$z_c = (z_1 + z_2 + z_3)\; x_N \qquad (11)$$
with
$$x_N = \frac{1}{\sqrt{(x_1 + x_2 + x_3)^2 + (y_1 + y_2 + y_3)^2 + (z_1 + z_2 + z_3)^2}}$$

The area of the 2 ni^2 triangles in a diamond can be calculated by (12) which is due to Huilier. The triangle sides are denoted by a, b and c. On the unit-sphere, the excess angle ε is equal to the area of the spherical triangle.

$$\tan\frac{\varepsilon}{4} = \sqrt{\tan\frac{s}{2}\tan\frac{s-a}{2}\tan\frac{s-b}{2}\tan\frac{s-c}{2}}$$
with $\qquad (12)$
$$s = \frac{1}{2}\,(a + b + c)$$

Since each grid-point is surrounded by six triangles (five triangles at the 12 special points), the grid-point is the centre of a hexagon (pentagon at the 12 special points) as is illustrated in Fig. 10. The co-ordinates of the vertices of the hexagon, i.e. the points Q_1, Q_2, ... , Q_6, are in a good approximation given by averaging the Cartesian co-ordinates of the three surrounding triangles vertices and normalizing to unit length, thus they follow from (11).

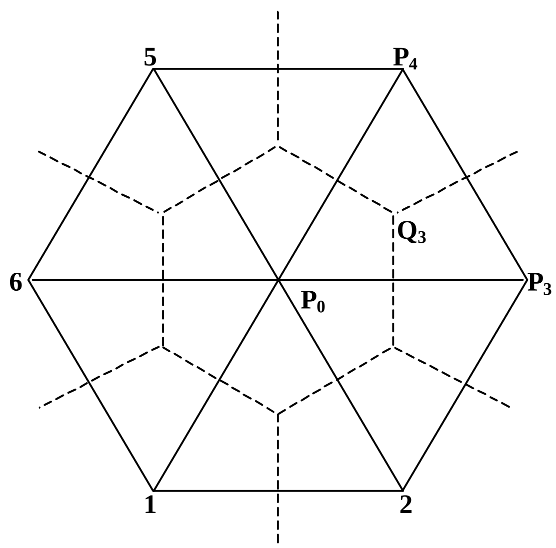

Figure 10 Hexagon connected to a grid-point of the triangular mesh

The grid-point in the centre of the hexagon is denoted by "0", the six surrounding triangles (and their vertices) by "1" to "6" counting counter-clockwise. We define the point Q_i, i.e. a vertex of the hexagon, equidistant from the three vertices P_0, P_i, and P_{i+1} such that Q_i and Q_{i+1} is the perpendicular bisection of the great circle P_0P_{i+1} (Fig. 10). The co-ordinates of Q_i are needed for the calculation of the topographical fields like orography, land fraction, roughness length as mean values over the area of the hexagons. Here, high-resolution data sets are averaged over the hexagon area.